

THE SINGULARITY CATEGORY OF A NAKAYAMA ALGEBRA

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ABSTRACT. Let A be a Nakayama algebra. We give a description of the singularity category of A inside its stable module category. We prove that there is a duality between the singularity category of A and the singularity category of its opposite algebra. As a consequence, the resolution quiver of A and the resolution quiver of its opposite algebra have the same number of cycles and the same number of cyclic vertices.

1. INTRODUCTION

Let A be an artin algebra. Denote by $A\text{-mod}$ the category of finitely generated left A -modules, and by $\mathbf{D}^b(A\text{-mod})$ the bounded derived category of $A\text{-mod}$. Recall that a complex in $\mathbf{D}^b(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex of finitely generated projective A -modules. Following [3, 11, 16], the *singularity category* $\mathbf{D}_{\text{sg}}(A)$ of A is the quotient triangulated category of $\mathbf{D}^b(A\text{-mod})$ with respect to the full subcategory consisting of perfect complexes. Recently, the singularity category of a Nakayama algebra is described in [7].

Let A be a connected Nakayama algebra without simple projective modules. Following [19], the *resolution quiver* $R(A)$ of A is defined as follows: the vertex set is the set of isomorphism classes of simple A -modules, and there is an arrow from S to $\tau \text{soc } P(S)$ for each simple A -module S ; see also [9]. Here, $P(S)$ is the projective cover of S , ‘soc’ denotes the socle of a module, and $\tau = D \text{Tr}$ is the Auslander-Reiten translation [1]. A simple A -module is called *cyclic* provided that it lies in a cycle of $R(A)$.

The following consideration is inspired by [19]. Let A be a connected Nakayama algebra of infinite global dimension. Let \mathcal{S}_c be a complete set of pairwise non-isomorphic cyclic simple A -modules. Let \mathcal{X}_c be the set formed by indecomposable A -modules X such that $\text{top } X$ and $\tau \text{soc } X$ both belong to \mathcal{S}_c . Here, ‘top’ denotes the top of a module. Denote by \mathcal{F} the full subcategory of $A\text{-mod}$ whose objects are finite direct sums of objects in \mathcal{X}_c . It turns out that \mathcal{F} is a Frobenius abelian category, and it is equivalent to $A'\text{-mod}$ with A' a connected selfinjective Nakayama algebra. Denote by $\underline{\mathcal{F}}$ the stable category of \mathcal{F} modulo projective objects; it is a triangulated category by [10]. We emphasize that the stable category $\underline{\mathcal{F}}$ is a full subcategory of the stable module category $A\text{-}\underline{\text{mod}}$ of A .

The well-known result of [3, 11] describes the singularity category of a Gorenstein algebra A via the subcategory of $A\text{-}\underline{\text{mod}}$ formed by Gorenstein projective modules. Here, we recall that an artin algebra is *Gorenstein* if the injective dimension of the regular module is finite on both sides. In general, a Nakayama algebra is not Gorenstein [19, 7]. The following result describes the singularity category of a Nakayama algebra via the subcategory $\underline{\mathcal{F}}$ of $A\text{-}\underline{\text{mod}}$. For a Gorenstein Nakayama algebra, these two descriptions coincide; compare [19].

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Theorem 1.1. *Let A be a connected Nakayama algebra of infinite global dimension. Then the singularity category $\mathbf{D}_{\text{sg}}(A)$ and the stable category $\underline{\mathcal{F}}$ are triangle equivalent.*

Denote by $A\text{-inj}$ the category of finitely generated injective A -modules, and by $\mathbf{K}^b(A\text{-inj})$ the bounded homotopy category of $A\text{-inj}$. We view $\mathbf{K}^b(A\text{-inj})$ as a thick subcategory of $\mathbf{D}^b(A\text{-mod})$ via the canonical functor. Then the quotient triangulated category $\mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-inj})$ is triangle equivalent to the opposite category of the singularity category $\mathbf{D}_{\text{sg}}(A^{\text{op}})$ of A^{op} . Here, A^{op} is the opposite algebra of A . In general, it seems that for an arbitrary artin algebra A , there is no obvious relation between $\mathbf{D}_{\text{sg}}(A)$ and $\mathbf{D}_{\text{sg}}(A^{\text{op}})$. However, we have the following result for a Nakayama algebra.

Proposition 1.2. *Let A be a Nakayama algebra. Then the singularity category $\mathbf{D}_{\text{sg}}(A)$ is triangle equivalent to $\mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-inj})$. Equivalently, there is a triangle duality between $\mathbf{D}_{\text{sg}}(A)$ and $\mathbf{D}_{\text{sg}}(A^{\text{op}})$.*

Let A be a connected Nakayama algebra of infinite global dimension. Recall from [20] that the resolution quivers $R(A)$ and $R(A^{\text{op}})$ have the same number of cyclic vertices. The following result strengthens the previous one by a different method.

Proposition 1.3. *Let A be a connected Nakayama algebra of infinite global dimension. Then the resolution quivers $R(A)$ and $R(A^{\text{op}})$ have the same number of cycles and the same number of cyclic vertices.*

The paper is organized as follows. In Section 2, we recall some facts on singularity categories of artin algebras and the simplification in the sense of [18]. In Section 3, we introduce the Frobenius subcategory \mathcal{F} and prove Theorem 1.1. The proofs of Propositions 1.2 and 1.3 are given in Sections 4 and 5, respectively.

Throughout this paper, we fix a commutative artinian ring R . All categories, morphisms and functors are R -linear.

2. PRELIMINARIES

We first recall some facts on the singularity category of an artin algebra.

Let A be an artin algebra over R . Recall that $A\text{-mod}$ denotes the category of finitely generated left A -modules. Let $A\text{-proj}$ denote the full subcategory consisting of projective A -modules, and $A\text{-inj}$ the full subcategory consisting of injective A -modules. Denote by $A\text{-}\underline{\text{mod}}$ the projectively stable category of finitely generated A -modules; it is obtained from $A\text{-mod}$ by factoring out the ideal of all maps which factor through projective A -modules; see [1, IV.1].

Recall that for an A -module M , its *syzygy* $\Omega(M)$ is the kernel of its projective cover $P(M) \rightarrow M$. This gives rise to the *syzygy functor* $\Omega : A\text{-}\underline{\text{mod}} \rightarrow A\text{-}\underline{\text{mod}}$. Let $\Omega^0(M) = M$ and $\Omega^{i+1}(M) = \Omega(\Omega^i(M))$ for $i \geq 0$. Denote by $\Omega^i(A\text{-mod})$ the full subcategory of $A\text{-mod}$ formed by modules M such that there is an exact sequence $0 \rightarrow M \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0$ with each P_j projective. We also denote by $\Omega_0^i(A\text{-mod})$ the full subcategory of $\Omega^i(A\text{-mod})$ formed by modules without indecomposable projective direct summands.

Recall that $\mathbf{D}^b(A\text{-mod})$ denotes the bounded derived category of $A\text{-mod}$, whose translation functor is denoted by $[1]$. For each integer n , let $[n]$ denote the n -th power of $[1]$. The category $A\text{-mod}$ is viewed as a full subcategory of $\mathbf{D}^b(A\text{-mod})$ by identifying an A -module with the corresponding stalk complex concentrated at degree zero. Recall that a complex in $\mathbf{D}^b(A\text{-mod})$ is *perfect* provided that it is isomorphic to a bounded complex of finitely generated projective A -modules. Perfect complexes form a thick subcategory of $\mathbf{D}^b(A\text{-mod})$, which is denoted by

$\mathbf{perf}(A)$. Here, we recall that a triangulated subcategory is *thick* if it is closed under direct summands.

Following [3, 11, 16], the quotient triangulated category

$$\mathbf{D}_{\text{sg}}(A) = \mathbf{D}^b(A\text{-mod})/\mathbf{perf}(A)$$

is called the *singularity category* of A . Denote by $q : \mathbf{D}^b(A\text{-mod}) \rightarrow \mathbf{D}_{\text{sg}}(A)$ the quotient functor. We recall that the objects in $\mathbf{D}_{\text{sg}}(A)$ are bounded complexes of finitely generated A -modules. The translation functor of $\mathbf{D}_{\text{sg}}(A)$ is also denoted by $[1]$.

The following results are well known.

Lemma 2.1 ([5, Lemma 2.1]). *Let X be a complex in $\mathbf{D}_{\text{sg}}(A)$ and $n_0 > 0$. Then for any n sufficiently large, there exists a module M in $\Omega^{n_0}(A\text{-mod})$ such that $X \simeq q(M)[n]$.*

Lemma 2.2 ([5, Lemma 2.2]). *Let $0 \rightarrow N \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$ be an exact sequence in $A\text{-mod}$ with each P_i projective. Then there is an isomorphism $q(M) \simeq q(N)[n]$ in $\mathbf{D}_{\text{sg}}(A)$. In particular, there is a natural isomorphism $\theta_M^n : q(M) \simeq q(\Omega^n(M))[n]$ for any M in $A\text{-mod}$ and $n \geq 0$.*

Observe that the composition $A\text{-mod} \rightarrow \mathbf{D}^b(A\text{-mod}) \xrightarrow{q} \mathbf{D}_{\text{sg}}(A)$ vanishes on projective modules. Then it induces a unique functor $q' : A\text{-mod} \rightarrow \mathbf{D}_{\text{sg}}(A)$. It follows from Lemma 2.2 that for each $n \geq 0$, the following diagram of functors

$$\begin{array}{ccc} A\text{-mod} & \xrightarrow{\Omega^n} & A\text{-mod} \\ q' \downarrow & & \downarrow q' \\ \mathbf{D}_{\text{sg}}(A) & \xrightarrow{[-n]} & \mathbf{D}_{\text{sg}}(A) \end{array}$$

is commutative. Let M and N be in $A\text{-mod}$ and $n \geq 0$. Lemma 2.2 yields a natural map

$$\Phi^n : \underline{\text{Hom}}_A(\Omega^n(M), \Omega^n(N)) \longrightarrow \text{Hom}_{\mathbf{D}_{\text{sg}}(A)}(q(M), q(N)).$$

Here, Φ^0 is induced by q' and $\Phi^n(f) = (\theta_N^n)^{-1} \circ (q'(f)[n]) \circ \theta_M^n$ for $n \geq 1$.

Consider the following chain of maps $\{G^{n,n+1}\}_{n \geq 0}$ such that

$$G^{n,n+1} : \underline{\text{Hom}}_A(\Omega^n(M), \Omega^n(N)) \longrightarrow \underline{\text{Hom}}_A(\Omega^{n+1}(M), \Omega^{n+1}(N))$$

is induced by the syzygy functor Ω . The sequence $\{\Phi^n\}_{n \geq 0}$ is compatible with $\{G^{n,n+1}\}_{n \geq 0}$, that is, $\Phi^{n+1} \circ G^{n,n+1} = \Phi^n$ for each $n \geq 0$. Then we obtain an induced map

$$\Phi : \varinjlim_{n \geq 0} \underline{\text{Hom}}_A(\Omega^n(M), \Omega^n(N)) \longrightarrow \text{Hom}_{\mathbf{D}_{\text{sg}}(A)}(q(M), q(N)).$$

Lemma 2.3 ([13, Exemple 2.3]). *Let M and N be in $A\text{-mod}$. Then there is a natural isomorphism*

$$\Phi : \varinjlim_{n \geq 0} \underline{\text{Hom}}_A(\Omega^n(M), \Omega^n(N)) \xrightarrow{\simeq} \text{Hom}_{\mathbf{D}_{\text{sg}}(A)}(q(M), q(N)).$$

Next we recall the *simplification* in the sense of [18].

Let \mathcal{A} be an abelian category. Recall that an object X in \mathcal{A} is a *brick* if $\text{End}_{\mathcal{A}}(X)$ is a division ring. Two objects X and Y are *orthogonal* if $\text{Hom}_{\mathcal{A}}(X, Y) = 0$ and $\text{Hom}_{\mathcal{A}}(Y, X) = 0$. A full subcategory \mathcal{W} of \mathcal{A} is called a *wide subcategory* if it is closed under kernels, cokernels and extensions. In particular, \mathcal{W} is an abelian category and the inclusion functor is exact. Recall that an abelian category \mathcal{A} is called a *length category* provided that each object in \mathcal{A} has a composition series.

Let \mathcal{E} be a set of objects in an abelian category \mathcal{A} . For an object C in \mathcal{A} , an \mathcal{E} -filtration of C is given by a sequence of subobjects

$$0 = C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots \subseteq C_m = C,$$

such that each factor C_i/C_{i-1} belongs to \mathcal{E} for $1 \leq i \leq m$. Denote by $\mathcal{F}(\mathcal{E})$ the full subcategory of \mathcal{A} formed by objects in \mathcal{A} with an \mathcal{E} -filtration.

Lemma 2.4 ([18, Theorem 1.2]). *Let \mathcal{E} be a set of pairwise orthogonal bricks in \mathcal{A} . Then $\mathcal{F}(\mathcal{E})$ is a wide subcategory of \mathcal{A} ; moreover, $\mathcal{F}(\mathcal{E})$ is a length category and \mathcal{E} is a complete set of pairwise non-isomorphic simple objects in $\mathcal{F}(\mathcal{E})$.*

Let A be a connected Nakayama algebra without simple projective modules. Recall that the vertex set of the *resolution quiver* $R(A)$ of A is the set of isomorphism classes of simple A -modules, and there is an arrow from S to $\gamma(S) = \tau \operatorname{soc} P(S)$ for each simple A -module S . Since each vertex in $R(A)$ is the start of a unique arrow, each connected component of $R(A)$ contains precisely one cycle. A simple A -module is called *cyclic* provided that it lies in a cycle of $R(A)$.

Let A be a connected Nakayama algebra of infinite global dimension. In particular, A has no simple projective modules. Let \mathcal{S} be a complete set of pairwise non-isomorphic simple A -modules. Denote by \mathcal{S}_c the subset of all cyclic simple A -modules, and by \mathcal{S}_{nc} the subset of all *noncyclic* simple A -modules.

Lemma 2.5. *Let A be a connected Nakayama algebra of infinite global dimension. Then \mathcal{S}_c is a complete set of pairwise non-isomorphic simple A -modules of infinite injective dimension, and \mathcal{S}_{nc} is a complete set of pairwise non-isomorphic simple A -modules of finite injective dimension.*

Proof. This is dual to [14, Corollaries 3.6 and 3.7]. \square

We will need the following fact. Recall that ‘top’ denotes the top of a module.

Lemma 2.6 ([19, Corollary to Lemma 2]). *Let A be a connected Nakayama algebra without simple projective modules. Assume that M is an indecomposable A -module and $m \geq 0$. Then either $\Omega^{2m}(M) = 0$ or else $\operatorname{top} \Omega^{2m}(M) = \gamma^m(\operatorname{top} M)$.*

3. A FROBENIUS SUBCATEGORY

In this section, we introduce a Frobenius subcategory in the module category of a Nakayama algebra, whose stable category is triangle equivalent to the singularity category of the given algebra.

Throughout this section, A is a connected Nakayama algebra of infinite global dimension. Denote by $n(A)$ the number of the isomorphism classes of simple A -modules. Denote by $l(M)$ the composition length of an A -module M . Recall that \mathcal{S} denotes a complete set of pairwise non-isomorphic simple A -modules, \mathcal{S}_c the subset of all cyclic simple A -modules and \mathcal{S}_{nc} the subset of all noncyclic simple A -modules. Observe that the map γ restricts to a permutation on \mathcal{S}_c . Let \mathcal{X}_c be the set formed by indecomposable A -modules X such that both $\operatorname{top} X$ and $\tau \operatorname{soc} X$ belong to \mathcal{S}_c .

The proof of the following well-known lemma uses the structure of indecomposable modules over Nakayama algebras; see [1, IV.3 and VI.2]. Each indecomposable A -module X is uniserial, and it is uniquely determined by its top and its composition length. Its composition factors from the top are $S, \tau S, \dots, \tau^{l-1} S$, where $S = \operatorname{top} X$ and $l = l(X)$. In particular, the projective cover $P(X)$ of X is indecomposable.

Lemma 3.1. *Let M be an indecomposable A -module which contains a nonzero projective submodule P . Then M is projective.*

Proof. Suppose that, on the contrary, M is nonprojective. Then there is a proper surjective map $\pi : P(M) \rightarrow M$, where $P(M)$ is the projective cover of M . We have a proper surjective map $\pi^{-1}(P) \rightarrow P$; it splits, since P is projective. This is impossible, since $P(M)$ is indecomposable, and thus its submodule $\pi^{-1}(P)$ is indecomposable. \square

Lemma 3.2. *Let $f : X \rightarrow Y$ be a morphism in \mathcal{X}_c . Then $\text{Ker } f$, $\text{Coker } f$ and $\text{Im } f$ belong to $\mathcal{X}_c \cup \{0\}$.*

Proof. We may assume that f is nonzero. Then we have $\text{top}(\text{Im } f) = \text{top } X$ and $\tau \text{soc}(\text{Im } f) = \tau \text{soc } Y$, both of which belong to \mathcal{S}_c . Thus, $\text{Im } f$ belongs to \mathcal{X}_c .

If f is not a monomorphism, then $\text{top}(\text{Ker } f) = \tau \text{soc}(\text{Im } f) = \tau \text{soc } Y$ and $\tau \text{soc}(\text{Ker } f) = \tau \text{soc } X$. Thus, $\text{Ker } f$ belongs to \mathcal{X}_c .

If f is not an epimorphism, then $\text{top}(\text{Coker } f) = \text{top } Y$ and $\tau \text{soc}(\text{Coker } f) = \text{top}(\text{Im } f) = \text{top } X$. Thus, $\text{Coker } f$ belongs to \mathcal{X}_c . \square

Lemma 3.3. *Let X be an object in \mathcal{X}_c . If $0 \subsetneq X'' \subsetneq X' \subsetneq X$ are subobjects of X such that X'/X'' belongs to \mathcal{X}_c , then X'' and X/X' belong to \mathcal{X}_c .*

Proof. Since both $\text{top } X'' = \tau \text{soc } X'/X''$ and $\tau \text{soc } X'' = \tau \text{soc } X$ belong to \mathcal{S}_c , it follows from the definition that X'' belongs to \mathcal{X}_c . Similarly, since both $\text{top } X/X' = \text{top } X$ and $\tau \text{soc } X/X' = \text{top } X'/X''$ belong to \mathcal{S}_c , it follows from the definition that X/X' belongs to \mathcal{X}_c . \square

Denote by \mathcal{P}_c a complete set of projective covers of modules in \mathcal{S}_c . We claim that \mathcal{P}_c is a subset of \mathcal{X}_c . Indeed, we have $\text{top } P(S) = S$ and $\tau \text{soc } P(S) = \gamma(S)$, both of which belong to \mathcal{S}_c . It follows that \mathcal{X}_c is closed under projective covers.

For each S in \mathcal{S}_c , let $E(S)$ denote the indecomposable A -module of the least composition length among those objects X in \mathcal{X}_c with $\text{top } X = S$. Inspired by [19, Section 4], we call $E(S)$ the *elementary module* associated to S . Denote by \mathcal{E}_c the set of elementary modules. Recall that $\mathcal{F}(\mathcal{E}_c)$ is the full subcategory of $A\text{-mod}$ formed by A -modules with an \mathcal{E}_c -filtration.

The *support* of an A -module M is the subset of \mathcal{S} consisting of those simple A -modules appearing as a composition factor of M . For a set \mathcal{X} of A -modules, we denote by $\text{add } \mathcal{X}$ the full subcategory of $A\text{-mod}$ whose objects are direct summands of finite direct sums of objects in \mathcal{X} .

The following result is in spirit close to [19, Proposition 2]. In particular, we prove that each elementary module E is a brick and thus $l(E) \leq n(A)$.

Proposition 3.4. *Let A be a connected Nakayama algebra of infinite global dimension. Then the following statements hold.*

- (1) *The set \mathcal{E}_c of elementary modules is a set of pairwise orthogonal bricks, and thus $\mathcal{F}(\mathcal{E}_c)$ is a wide subcategory of $A\text{-mod}$.*
- (2) *$\mathcal{F}(\mathcal{E}_c) = \text{add } \mathcal{X}_c$, which is closed under projective covers.*
- (3) *Let E and E' be elementary modules. Then $E = E'$ if and only if their supports have nonempty intersection.*

Proof. (1) Let $f : E \rightarrow E'$ be a nonzero map between elementary modules. By Lemma 3.2 $\text{Im } f$ belongs to \mathcal{X}_c . However, $\text{Im } f$ is a factor module of E . By the definition of the elementary module E we have $E = \text{Im } f$. Then f is an injective map. Similarly, f is a surjective map and thus an isomorphism. Therefore \mathcal{E}_c is a set of pairwise orthogonal bricks.

By Lemma 2.4 $\mathcal{F}(\mathcal{E}_c)$ is a wide subcategory of $A\text{-mod}$. In particular, it is closed under direct sums and direct summands.

(2) We prove that any module X in \mathcal{X}_c belongs to $\mathcal{F}(\mathcal{E}_c)$, and thus $\text{add } \mathcal{X}_c \subseteq \mathcal{F}(\mathcal{E}_c)$. We use induction on $l(X)$. Set $S = \text{top } X \in \mathcal{S}_c$. If $X = E(S) \in \mathcal{E}_c$, we are

done. Otherwise, there is a proper surjective map $\pi : X \rightarrow E(S)$. By Lemma 3.2 we have $\text{Ker } \pi \in \mathcal{X}_c$. Then by induction $\text{Ker } \pi \in \mathcal{F}(\mathcal{E}_c)$. Therefore $X \in \mathcal{F}(\mathcal{E}_c)$.

Recall that each elementary module E satisfies that $\text{top } E \in \mathcal{S}_c$ and $\tau \text{soc } E \in \mathcal{S}_c$. It follows from its \mathcal{E}_c -filtration that each indecomposable object X in $\mathcal{F}(\mathcal{E}_c)$ satisfies that $\text{top } X \in \mathcal{S}_c$ and $\tau \text{soc } X \in \mathcal{S}_c$. Then by definition X belongs to \mathcal{X}_c . Therefore $\text{add } \mathcal{X}_c \supseteq \mathcal{F}(\mathcal{E}_c)$, and thus $\mathcal{F}(\mathcal{E}_c) = \text{add } \mathcal{X}_c$. Since \mathcal{X}_c is closed under projective covers, we infer that $\text{add } \mathcal{X}_c$ is closed under projective covers.

(3) Suppose that $E \neq E'$ have a common composition factor. We may assume that $l(E) \leq l(E')$. Since E and E' are orthogonal, there exists a chain $0 \subsetneq E_1 \subsetneq E_2 \subsetneq E'$ of A -modules such that $E_2/E_1 = E$. By Lemma 3.3 we have that E'/E_2 belongs to \mathcal{X}_c . This contradicts to the definition of the elementary module E' . \square

Lemma 3.5. *Let S be a cyclic simple A -module. Then the following statements hold.*

- (1) *The injective dimension of $E(S)$ is infinite, and the injective dimension of $P(S)$ is finite.*
- (2) *There is a unique simple A -module S' in \mathcal{S}_c such that $\text{top } E(S') = \tau \text{soc } E(S)$ and $\text{Ext}_A^1(E(S), E(S')) \neq 0$.*

Proof. (1) We recall from Lemma 2.5 that \mathcal{S}_c is a complete set of pairwise non-isomorphic simple A -modules of infinite injective dimension. Since the elementary modules have pairwise disjoint supports, for each S in \mathcal{S}_c , the support of $E(S)$ contains precisely one cyclic simple A -module, that is, S . In other words, each composition factor of $E(S)$ different from S is a noncyclic simple A -module, and thus has finite injective dimension. It follows that $E(S)$ has infinite injective dimension.

Let $h : P(S) \rightarrow I$ be an injective envelope of the A -module $P(S)$. We claim that each composition factor S' of $\text{Coker } h$ is a noncyclic simple A -module, and thus has finite injective dimension. Consequently, the injective dimension of $\text{Coker } h$ is finite. Therefore the injective dimension of $P(S)$ is finite.

For the claim, we observe by Lemma 3.1 $P(S) \subsetneq P(S') \subseteq I$. Then we have $\gamma(S') = \gamma(S)$. Recall that the restriction of γ on cyclic simple A -modules is injective. Therefore S' is a noncyclic simple A -module, since S is a cyclic simple A -module and $S' \neq S$.

(2) Let $E = E(S)$. Recall that $P(S)$ lies in \mathcal{X}_c and thus in $\mathcal{F}(\mathcal{E}_c)$. Consider the \mathcal{E}_c -filtration of $P(S)$, say

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{t-1} \subsetneq M_t = P(S),$$

such that M_i/M_{i-1} is elementary for $1 \leq i \leq t$. We observe that $M_t/M_{t-1} = E$ and $t \geq 2$, since by (1) we have $E(S) \neq P(S)$. Set $E' = M_{t-1}/M_{t-2}$. Note that $E' = E(S')$ for some cyclic simple A -module S' . Then

$$\text{top } E' = \text{top}(M_{t-1}/M_{t-2}) = \tau \text{soc}(M_t/M_{t-1}) = \tau \text{soc } E.$$

Since $M_t/M_{t-2} = P(S)/M_{t-2}$ is indecomposable, the exact sequence

$$0 \rightarrow M_{t-1}/M_{t-2} \rightarrow M_t/M_{t-2} \rightarrow M_t/M_{t-1} \rightarrow 0$$

does not split. Then we have $\text{Ext}_A^1(E, E') \neq 0$. The uniqueness of S' is obvious, since $S' = \tau \text{soc } E(S)$. \square

Recall that by definition $\tau \text{soc } E$ lies in \mathcal{S}_c for each elementary module E . We have a map $\delta : \mathcal{S}_c \rightarrow \mathcal{S}_c$, which sends a cyclic simple A -module S to $\delta(S) = \tau \text{soc } E(S)$. We claim that δ is injective and thus bijective. Indeed, if $\delta(S) = \delta(\bar{S})$, then $\text{soc } E(S) = \text{soc } E(\bar{S})$. It follows from Lemma 3.4(3) that $S = \bar{S}$.

Corollary 3.6. *Let S be a cyclic simple A -module and t the minimal positive integer such that $\delta^t(S) = S$. Then $\mathcal{S}_c = \{S, \delta(S), \dots, \delta^{t-1}(S)\}$ and \mathcal{S} is the disjoint union of the supports of all elementary modules.*

Proof. Since A is a connected Nakayama algebra without simple projective modules, any nonempty subset of \mathcal{S} which is closed under τ must be \mathcal{S} . By Lemma 3.5(2) the union of the supports of all $E(\delta^i(S))$ is closed under τ , we infer that this union is \mathcal{S} . Let S' be a cyclic simple A -module. Then there exists an integer $0 \leq i \leq t-1$ such that the support of $E(\delta^i(S))$ and the support of $E(S)$ have nonempty intersection. It follows from Lemma 3.4(3) that $S' = \delta^i(S)$. \square

Proposition 3.7. *Let A be a connected Nakayama algebra of infinite global dimension. Then $\mathcal{F}(\mathcal{E}_c)$ is equivalent to A' -mod, where A' is a connected selfinjective Nakayama algebra.*

Proof. Let $P = \bigoplus_{S \in \mathcal{S}_c} P(S)$ and $A' = \text{End}_A(P)^{\text{op}}$. Then P is a projective object in $\mathcal{F}(\mathcal{E}_c)$, since $\mathcal{F}(\mathcal{E}_c)$ is a wide subcategory of A -mod. The natural projection $P(S) \rightarrow E(S)$ is a projective cover in the category $\mathcal{F}(\mathcal{E}_c)$. Recall from Lemma 2.4 that $\mathcal{F}(\mathcal{E}_c)$ is a length category with $\mathcal{E}_c = \{E(S) \mid S \in \mathcal{S}_c\}$ a complete set of pairwise non-isomorphic simple objects. We infer that for each object X in $\mathcal{F}(\mathcal{E}_c)$, there is an epimorphism $P' \rightarrow X$ with P' in $\text{add } P$. Then P is a projective generator for $\mathcal{F}(\mathcal{E}_c)$. We have an equivalence $\mathcal{F}(\mathcal{E}_c) \simeq A'$ -mod; compare [15, Chapter IV, Theorem 5.3].

Since each indecomposable object in $\mathcal{F}(\mathcal{E}_c)$ is uniserial, we infer that A' is a Nakayama algebra. Denote by τ' the Auslander-Reiten translation of A' . Then we have $\tau'E(S) = E(\delta(S))$ by Lemma 3.5(2). It follows from Corollary 3.6 that all simple A' -modules are in the same τ' -orbit. Therefore, the Nakayama algebra A' is connected.

It remains to show that A' is selfinjective. Since γ restricts to a permutation on \mathcal{S}_c , the modules in \mathcal{P}_c have pairwise distinct socles. Therefore, we have $\mathcal{S}_c = \{\tau \text{ soc } P \mid P \in \mathcal{P}_c\}$. Let E be an elementary module. Since $\tau \text{ soc } E$ lies in \mathcal{S}_c , there exists P in \mathcal{P}_c with $\text{soc } P = \text{soc } E$ and $\text{soc } P \neq \text{soc } E'$ for any elementary module $E' \neq E$. It follows that the socle of P in the category $\mathcal{F}(\mathcal{E}_c)$ is E . We have proven that every simple A' -module can embed into a projective A' -module. Therefore, A' is selfinjective. \square

The following result is analogous to [19, Proposition 4].

Lemma 3.8. *The following statements are equivalent for an indecomposable nonprojective A -module M .*

- (1) M belongs to $\mathcal{F}(\mathcal{E}_c)$.
- (2) There is an exact sequence $0 \rightarrow M \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ for some $n \geq 1$ such that each P_i belongs to \mathcal{P}_c .
- (3) There is an exact sequence $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ such that P_i belongs to \mathcal{P}_c for $i = 0, 1$.

Proof. “(1) \Rightarrow (2)” Recall that for an indecomposable nonprojective module M' over a selfinjective Nakayama algebra A' , there exists an exact sequence $0 \rightarrow M' \rightarrow P'_n \rightarrow \dots \rightarrow P'_1 \rightarrow P'_0 \rightarrow M' \rightarrow 0$ for some $n \geq 1$ such that each P'_i is indecomposable projective. Then (2) follows from Proposition 3.7.

“(2) \Rightarrow (3)” This is obvious.

“(3) \Rightarrow (1)” Observe that $\text{top } M = \text{top } P_0$ and $\tau \text{ soc } M = \text{top } \Omega M = \text{top } P_1$, both of which belong to \mathcal{S}_c . Then by definition M belongs to \mathcal{X}_c . \square

Recall that each component of the resolution quiver $R(A)$ has a unique cycle. For each noncyclic vertex S in $R(A)$, there exists a unique path of minimal length starting with S and ending in a cycle. We call the length of this path the distance

between S and the cycle. Let $d(A)$ be the maximal distance between noncyclic vertices and cycles. Observe that $\gamma^d(S)$ is cyclic for each simple A -module S .

Lemma 3.9. *Let $d = d(A)$ be as above. Then the following statements hold.*

- (1) $\Omega^{2d}(M)$ belongs to $\mathcal{F}(\mathcal{E}_c)$ for any M in $A\text{-mod}$.
- (2) $\Omega_0^{2d}(A\text{-mod}) \subseteq \mathcal{F}(\mathcal{E}_c) \subseteq \Omega^{2d}(A\text{-mod})$.

Proof. (1) We may assume that M is indecomposable. It follows from Lemma 2.6 that either $\Omega^{2d}(M)$ is zero or $\text{top } \Omega^{2d}(M) = \gamma^d(\text{top } M)$. If $\Omega^{2d}(M)$ is indecomposable projective, then $\Omega^{2d}(M)$ belongs to \mathcal{P}_c . If $\Omega^{2d}(M)$ is indecomposable non-projective, then $\text{top } \Omega^{2d}(M) = \gamma^d(\text{top } M)$ and $\tau \text{soc } \Omega^{2d}(M) = \text{top } \Omega^{2d+1}(M) = \gamma^d(\text{top } \Omega M)$. Then by definition M belongs to \mathcal{X}_c .

(2) The first inclusion follows from (1), and the second one follows from Lemma 3.8(2). \square

By Proposition 3.7 $\mathcal{F}(\mathcal{E}_c)$ is a Frobenius category whose projective objects are precisely add \mathcal{P}_c . Denote by $\underline{\mathcal{F}}(\mathcal{E}_c)$ the stable category of $\mathcal{F}(\mathcal{E}_c)$ modulo projective objects. It is a triangulated category; see [10].

Recall from Proposition 3.4 that $\mathcal{F}(\mathcal{E}_c)$ is a wide subcategory of $A\text{-mod}$ which is closed under projective covers. Consider the inclusion functor $i : \mathcal{F}(\mathcal{E}_c) \rightarrow A\text{-mod}$. It induces uniquely a fully-faithful functor $i' : \underline{\mathcal{F}}(\mathcal{E}_c) \rightarrow A\text{-mod}$. We recall the induced functor $q' : A\text{-mod} \rightarrow \mathbf{D}_{\text{sg}}(A)$ in Section 2.

The following is the main result of this section, which describes the singularity category of A as a subcategory of the stable module category of A .

Theorem 3.10. *Let A be a connected Nakayama algebra of infinite global dimension. Then the composite functor $q' \circ i' : \underline{\mathcal{F}}(\mathcal{E}_c) \rightarrow A\text{-mod} \rightarrow \mathbf{D}_{\text{sg}}(A)$ is a triangle equivalence.*

Proof. Observe that the composite functor $\mathcal{F}(\mathcal{E}_c) \xrightarrow{i} A\text{-mod} \rightarrow \mathbf{D}^b(A\text{-mod}) \xrightarrow{q} \mathbf{D}_{\text{sg}}(A)$ is a ∂ -functor in the sense of [12, Section 1]; compare [4, Lemma 2.4]. Then the functor $q' \circ i'$ is a triangle functor; see [4, Lemma 2.5].

Recall that the subcategory $\mathcal{F}(\mathcal{E}_c)$ of $A\text{-mod}$ is wide and closed under projective covers; moreover, $\mathcal{F}(\mathcal{E}_c)$ is a Frobenius category. Then the restriction of the syzygy functor $\Omega : A\text{-mod} \rightarrow A\text{-mod}$ on $\mathcal{F}(\mathcal{E}_c)$ is an autoequivalence, in particular, it is fully faithful. Then the functor $q' \circ i'$ is fully faithful by the natural isomorphism in Lemma 2.3.

It remains to show that the functor $q' \circ i'$ is also dense. Let X be an object in $\mathbf{D}_{\text{sg}}(A)$. It follows from Lemmas 2.1 and 3.9(1) that there exists a module M in $\mathcal{F}(\mathcal{E}_c)$ and n sufficiently large such that $X \simeq q(M)[n]$ in $\mathbf{D}_{\text{sg}}(A)$. By above, the image $\text{Im}(q' \circ i')$ is a triangulated subcategory of $\mathbf{D}_{\text{sg}}(A)$, in particular, it is closed under $[m]$ for all $m \in \mathbb{Z}$. It follows from $X \simeq q(M)[n]$ that X lies in $\text{Im}(q' \circ i')$. This finishes our proof. \square

We observe the following immediate consequence of Proposition 3.7 and Theorem 3.10.

Corollary 3.11 (compare [7, Corollary 3.11]). *Let A be a connected Nakayama algebra of infinite global dimension. Then there is a triangle equivalence between $\mathbf{D}_{\text{sg}}(A)$ and $A'\text{-mod}$ for a connected selfinjective Nakayama algebra A' .*

4. A DUALITY BETWEEN SINGULARITY CATEGORIES

In this section, we prove that there is a triangle duality between the singularity category of a Nakayama algebra and the singularity category of its opposite algebra. The proof uses the Frobenius subcategory in the previous section.

Let A be a connected Nakayama algebra of infinite global dimension. We recall from Propositions 3.4 and 3.7 that the category $\mathcal{F} = \mathcal{F}(\mathcal{E}_c)$ is a wide subcategory of $A\text{-mod}$ closed under projective covers; it is equivalent to $A'\text{-mod}$ for a connected selfinjective Nakayama algebra A' .

Consider the inclusion functor $i : \mathcal{F} \rightarrow A\text{-mod}$. We claim that it admits an exact right adjoint $i_\rho : A\text{-mod} \rightarrow \mathcal{F}$.

For the claim, recall from the proof of Proposition 3.7 that $A' = \text{End}_A(P)^{\text{op}}$ with $P = \bigoplus_{S \in \mathcal{S}_c} P(S)$. We identify \mathcal{F} with $A'\text{-mod}$. Then the inclusion i is identified with $P \otimes_{A'} -$. The right adjoint is given by $i_\rho = \text{Hom}_A(P, -)$. It is exact since ${}_A P$ is projective.

The adjoint pair (i, i_ρ) induces an adjoint pair (i^*, i_ρ^*) of triangle functors between bounded derived categories. Here, for an exact functor F between abelian categories, F^* denotes its extension on bounded derived categories.

Recall that $\mathbf{K}^b(A\text{-inj})$ denotes the the bounded homotopy category of $A\text{-inj}$. We view $\mathbf{K}^b(A\text{-inj})$ as a thick subcategory of $\mathbf{D}^b(A\text{-mod})$ via the canonical functor. We mention that by the usual duality on module categories, the quotient triangulated category $\mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-inj})$ is triangle equivalent to the opposite category of the singularity category $\mathbf{D}_{\text{sg}}(A^{\text{op}})$ of A^{op} . Here, A^{op} is the opposite algebra of A .

The proof of the following result is similar to [7, Proposition 2.13]. Recall that \mathcal{P}_c is a complete set of pairwise non-isomorphic indecomposable projective objects in $\mathcal{F} = \mathcal{F}(\mathcal{E}_c)$.

Lemma 4.1. *Let A be a connected Nakayama algebra of infinite global dimension. Then the above functors i_ρ^* and i^* induce mutually inverse triangle equivalences between $\mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-inj})$ and $\mathbf{D}^b(\mathcal{F})/\mathbf{K}^b(\text{add } \mathcal{P}_c)$.*

Proof. Observe by [6, Lemma 3.3.1] that $i^* : \mathbf{D}^b(\mathcal{F}) \rightarrow \mathbf{D}^b(A\text{-mod})$ is fully faithful. It follows that its right adjoint i_ρ^* induces a triangle equivalence

$$\overline{i_\rho^*} : \mathbf{D}^b(A\text{-mod})/\text{Ker } i_\rho^* \simeq \mathbf{D}^b(\mathcal{F});$$

see [8, Chapter I, Section 1, 1.3 Proposition]. Here, $\text{Ker } F$ denotes the essential kernel of an additive functor F .

We claim that $\text{Ker } i_\rho^* = \text{thick}(\mathcal{S}_{nc})$, the smallest thick subcategory of $\mathbf{D}^b(A\text{-mod})$ containing \mathcal{S}_{nc} . Here, we recall that \mathcal{S}_{nc} denotes the set of noncyclic simple A -modules. By Lemma 2.5 each noncyclic simple A -module has finite injective dimension. It follows from the claim that $\text{Ker } i_\rho^* \subseteq \mathbf{K}^b(A\text{-inj})$.

For the claim, we observe that $\text{Ker } i_\rho = \mathcal{F}(\mathcal{S}_{nc})$, the full subcategory of $A\text{-mod}$ formed by A -modules with an \mathcal{S}_{nc} -filtration. The claim follows from the fact that a complex X is in $\text{Ker } i_\rho^*$ if and only if each cohomology $H^i(X)$ is in $\text{Ker } i_\rho$.

We observe that i_ρ preserves injective objects since it has an exact left adjoint. It follows that $i_\rho^*(\mathbf{K}^b(A\text{-inj})) \subseteq \mathbf{K}^b(\text{add } \mathcal{P}_c)$. By Lemma 3.5(1) each module Q in \mathcal{P}_c has finite injective dimension. Note that $i_\rho^* Q = Q$. Therefore $i_\rho^*(\mathbf{K}^b(A\text{-inj})) \supseteq \mathbf{K}^b(\text{add } \mathcal{P}_c)$, and thus $i_\rho^*(\mathbf{K}^b(A\text{-inj})) = \mathbf{K}^b(\text{add } \mathcal{P}_c)$. From this equality, the triangle equivalence $\overline{i_\rho^*}$ restricts to a triangle equivalence

$$\mathbf{K}^b(A\text{-inj})/\text{Ker } i_\rho^* \simeq \mathbf{K}^b(\text{add } \mathcal{P}_c).$$

The desired equivalence follows from [21, Chapitre I, §2, 4-3 Corollaire]. \square

Proposition 4.2. *Let A be a Nakayama algebra. Then the singularity category $\mathbf{D}_{\text{sg}}(A)$ is triangle equivalent to $\mathbf{D}^b(A\text{-mod})/\mathbf{K}^b(A\text{-inj})$. Equivalently, there is a triangle duality between $\mathbf{D}_{\text{sg}}(A)$ and $\mathbf{D}_{\text{sg}}(A^{\text{op}})$.*

Proof. Without loss of generality, we may assume that A is a connected Nakayama algebra of infinite global dimension. Then the singularity category $\mathbf{D}_{\text{sg}}(A)$ is triangle equivalent to the stable category $\underline{\mathcal{F}}$ by Theorem 3.10.

Since \mathcal{F} is a Frobenius abelian category, it follows from [17, Theorem 2.1] that the stable category $\underline{\mathcal{F}}$ is triangle equivalent to $\mathbf{D}^b(\mathcal{F})/\mathbf{K}^b(\text{add } \mathcal{P}_c)$. Then the conclusion follows from Lemma 4.1. \square

5. THE RESOLUTION QUIVERS

Let A be a connected Nakayama algebra of infinite global dimension. Recall that $n(A)$ denotes the number of isomorphism classes of simple A -modules. By Corollary 3.11 the Auslander-Reiten quiver of the singularity category $\mathbf{D}_{\text{sg}}(A)$ is isomorphic to a truncated tube $\mathbb{Z}\mathbb{A}_m/\langle\tau^t\rangle$, where $m = m(A)$ denotes its height and $t = t(A)$ denotes its rank. Here, we use the fact that the Auslander-Reiten quiver of the stable module category of a connected selfinjective Nakayama algebra is a truncated tube; compare [1, VI.2].

Recall that $R(A)$ denotes the resolution quiver of A . We denote by $c(A)$ the number of cycles in $R(A)$. Let C be a cycle in $R(A)$. Then the *size* $s(C)$ of C is the number of vertices in C , and the *weight* $w(C)$ of C is $\frac{\sum_S l(P(S))}{n(A)}$, where S runs through all vertices of C . Here, $l(P(S))$ is the composition length of the projective cover $P(S)$ of a simple A -module S . Recall from [20] that all cycles in the resolution quiver $R(A)$ have the same size and the same weight. We denote $s(A) = s(C)$ and $w(A) = w(C)$ for an arbitrary cycle C in $R(A)$.

For two positive integers a and b , we denote their greatest common divisor by (a, b) .

Lemma 5.1. *Let $m = m(A)$ and $t = t(A)$ be as above. Then $c(A) = (m + 1, t)$, $s(A) = \frac{t}{(m+1, t)}$ and $w(A) = \frac{m+1}{(m+1, t)}$.*

Proof. Recall from [7, Theorem 3.8] that there exists a sequence of algebra homomorphisms

$$A = A_0 \xrightarrow{\eta_0} A_1 \xrightarrow{\eta_1} A_2 \rightarrow \cdots \rightarrow A_{r-1} \xrightarrow{\eta_{r-1}} A_r$$

such that each A_i is a connected Nakayama algebra and A_r is selfinjective; moreover, each η_i induces a triangle equivalence between $\mathbf{D}_{\text{sg}}(A_i)$ and $\mathbf{D}_{\text{sg}}(A_{i+1})$. Following [20, Lemma 2.2], each η_i induces a bijection between the set of cycles in $R(A_i)$ and the set of cycles in $R(A_{i+1})$, which preserves sizes and weights. Then we have $m(A_i) = m(A_{i+1})$, $t(A_i) = t(A_{i+1})$, $c(A_i) = c(A_{i+1})$, $s(A_i) = s(A_{i+1})$ and $w(A_i) = w(A_{i+1})$ for $0 \leq i \leq r-1$. Therefore it is enough to prove the equations for selfinjective Nakayama algebras.

Let A be a connected selfinjective Nakayama algebra. Then t equals the number of isomorphism classes of simple A -modules, and $m + 1$ equals the radical length of A . We claim that $s(A) = \frac{t}{(m+1, t)}$. Therefore, we have $c(A) = \frac{t}{s(A)} = (m + 1, t)$ and $w(A) = \frac{(m+1)s(A)}{t} = \frac{m+1}{(m+1, t)}$.

For the claim, let $\{S_1, \dots, S_t\}$ be a complete set of pairwise non-isomorphic simple A -modules such that $\tau S_i = S_{i+1}$ for $1 \leq i \leq t$. Here, we let $S_{t+j} = S_j$ for each $j > 0$. Then we have $\gamma(S_i) = S_{i+m+1}$, and thus $\gamma^d(S_i) = S_i$ if and only if t divides $d(m + 1)$. It follows that $R(A)$ consists of cycles of size $\frac{t}{(m+1, t)}$. \square

The following result establishes the relationship between the resolution quiver of a Nakayama algebra and the resolution quiver of its opposite algebra.

Proposition 5.2. *Let A be a connected Nakayama algebra of infinite global dimension. Then the following statements hold.*

- (1) *The resolution quivers $R(A)$ and $R(A^{\text{op}})$ have the same number of cycles and the same number of cyclic vertices.*
- (2) *All cycles in $R(A)$ and $R(A^{\text{op}})$ have the same weight.*

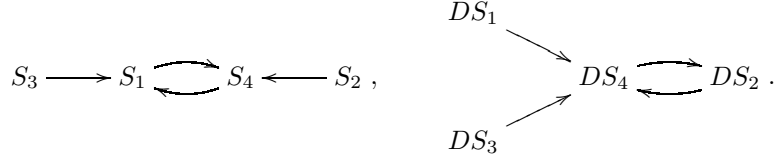
Proof. By Proposition 4.2 there is a triangle duality between $\mathbf{D}_{\text{sg}}(A)$ and $\mathbf{D}_{\text{sg}}(A^{\text{op}})$. Then the Auslander-Reiten quiver of $\mathbf{D}_{\text{sg}}(A^{\text{op}})$ is isomorphic to the opposite quiver of the Auslander-Reiten quiver of $\mathbf{D}_{\text{sg}}(A)$. Therefore, we have $m(A) = m(A^{\text{op}})$ and $t(A) = t(A^{\text{op}})$. It follows from Lemma 5.1 that $c(A) = c(A^{\text{op}})$, $s(A) = s(A^{\text{op}})$ and $w(A) = w(A^{\text{op}})$. \square

The following example shows that these two resolution quivers $R(A)$ and $R(A^{\text{op}})$ may not be isomorphic in general.

Example 5.3. Let A be a connected Nakayama algebra with admissible sequence $(7, 6, 6, 5)$. Assume that $\{S_1, S_2, S_3, S_4\}$ is a complete set of pairwise non-isomorphic simple A -modules such that $\tau S_i = S_{i+1}$ for $1 \leq i \leq 4$. Then we have $l(P_1) = 7$, $l(P_2) = 6$, $l(P_3) = 6$ and $l(P_4) = 5$. There is an arrow from S_i to S_j in $R(A)$ if and only if 4 divides $i - j + l(P_i)$.

Denote by D the usual duality, and by $(-)^*$ the duality on projectives. Then $\{DS_4, DS_3, DS_2, DS_1\}$ is a complete set of pairwise non-isomorphic simple A^{op} -modules such that $\tau' DS_i = DS_{i-1}$. Here, τ' is the Auslander-Reiten translation of A^{op} . We observe that $l(P_4^*) = 6$, $l(P_3^*) = 7$, $l(P_2^*) = 6$ and $l(P_1^*) = 5$. Therefore the admissible sequence of A^{op} is $(6, 7, 6, 5)$. There is an arrow from DS_i to DS_j in $R(A^{\text{op}})$ if and only if 4 divides $i - j - l(P_i^*)$.

The resolution quivers $R(A)$ and $R(A^{\text{op}})$ are shown as follows.



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